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**The photon spectrum in orthopositronium decay at**

$$\omega_\gamma \ll m_e$$

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**Abstract**

A closed analytic expression is given for the spectrum of low energy photons in the annihilation of orthopositronium, which expression sums all the effects of the Coulomb interaction between the electron and the positron. The applicability of the formula is limited only by the condition  $\omega_\gamma \ll m_e$ . In the region  $\omega_\gamma \gg m_e \alpha^2$  the Coulomb interaction term gives the leading at low energy one-loop correction, proportional to  $\alpha \sqrt{m_e/\omega_\gamma}$ , to the decay spectrum. The constant in  $\omega_\gamma$  radiative term in the one-loop correction to the spectrum is also presented here in an analytic form.

# 1 Introduction

The interplay of binding and radiative effects in the properties of positronium makes it an interesting case study in pure QED. In particular, the annihilation of the lowest  $^3S_1$  state of orthopositronium (o-Ps) into three photons has been studied both experimentally and theoretically since the first calculation of this process by Ore and Powell[1]. Most recently Manohar and Ruiz-Femenía[2] have considered the effects of the Coulomb interaction on the photon spectrum in the o-Ps decay in the region of small photon energies, where these effects are essential. They have found in an integral form the expression for this spectrum, which takes into account to all orders the Coulomb interaction between the electron and the positron in the nonrelativistic approximation, and they also argued that their result is applicable in the region of the photon energy,  $\omega$ , satisfying the condition  $\omega \ll m\alpha$ , with  $m$  being the mass of the electron. The present paper contains essentially two further developments over the results of Ref.[2]: a closed analytic formula in terms of just one standard hypergeometric function is given for the spectrum of low energy photons, and it is shown that the condition for its applicability is in fact  $\omega \ll m$ , i.e. the formula is valid in a significantly broader range of energies than stated in Ref.[2]. Furthermore, in the region  $\omega \gg m\alpha^2$ , the expansion parameter for the Coulomb effects is  $\alpha\sqrt{m/\omega}$ , and the first term of this expansion describes the low-energy asymptotic behavior of the one-loop QED corrections to the spectrum. Due to the rather slow rise of the Coulomb term at low energy it is interesting to also evaluate the constant in  $\omega$  term at small  $\omega$  in the one-loop correction. As will be shown here this term can in fact be found from the known in the literature radiative QED corrections to the total rates of decay of the  $C$  even  $^1S_0$  and  $^3P_{0,2}$  states of positronium into two photons. A comparison of the resulting expression for the first two terms of the expansion in  $\omega/m$  of the  $O(\alpha)$  correction to the spectrum in the orthopositronium decay with the only available results of a numerical calculation[3] of this correction is also discussed here.

## 2 Multipole expansion at $\omega \ll m$

The spectrum of low-energy photons at  $\omega \ll m$  lends itself to a nonrelativistic treatment<sup>1</sup>. Indeed, after emission of such soft photon from the initial  $^3S_1$  state of o-Ps at energy  $E_0 =$

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<sup>1</sup>A similar treatment in QCD was used[4] in discussion of soft gluons in a three-gluon annihilation of heavy quarkonium.

$-m\alpha^2/4$  (relative to the threshold), the (virtual)  $e^+e^-$  pair remains nonrelativistic and is in a  $C$ -even state and has negative energy  $E = E_0 - \omega$ , which state then annihilates into two hard photons. This picture can be represented[4, 2] by the diagram (of the nonrelativistic perturbation theory) shown in Fig.1. The annihilation into two hard gluons takes place at distances  $O(m^{-1})$ , which are considered as infinitesimal in terms of the nonrelativistic relative motion of the electron and the positron. The Green's function at energy  $E$ , describing the propagation of the pair between the emission of the soft photon and annihilation, thus contains the exponential factor  $\exp(-\kappa r)$  with  $\kappa$  defined as  $-\kappa^2/m = E$ , so that the discussed process is determined by distances of order  $\kappa^{-1}$  and the typical velocities of the electron and positron are given by  $v \sim \kappa/m$ . It is well known that in such situation the expansion parameter for the Coulomb interaction is  $\alpha m/\kappa$ . Thus in the region where  $\kappa$  is comparable with  $m\alpha$  one has to use the exact Green's function in the Coulomb potential. In the region of larger  $\kappa$ :  $\kappa \gg m\alpha$ , the nonrelativistic treatment is still applicable as long as  $\kappa^2 \ll m^2$ , and the Coulomb effects can be calculated either by a perturbative expansion of the Green's function, or by an expansion of the exact formula, if such formula is available.

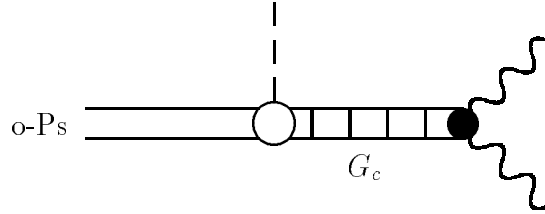


Figure 1: The diagram for the description of the photon spectrum in the region  $\omega \ll m$  in the decay of o-Ps. The open circle stands for the interaction of a nonrelativistic  $e^+e^-$  pair with a soft photon (dashed line), and the filled circle shows the annihilation into two hard photons (wavy lines) at distances  $O(m^{-1})$ .  $G_c$  stands for the Green's function in the Coulomb field.

The Hamiltonian for the emission of the soft photon in the diagram of Fig.1 can be expanded in multipoles. The dominance of the lowest multipoles is guaranteed by a general consideration[5] for as long as the system is nonrelativistic, i.e.  $\omega \ll m$ . However, since

at this point the present paper differs from Ref.[2], it is appropriate to provide here a more detailed discussion. The  $C$ -even state in Fig.1 annihilating into two hard gluons can be either a spin-triplet state with an odd orbital momentum  $L = 2n + 1$ , or a spin-singlet state with an even orbital momentum  $L = 2n$ , where in both cases  $n$  is a non-negative integer. For the first set of states the minimal multipole contributing to the transition  ${}^3S \rightarrow {}^3(2n + 1) + \gamma$  is of the electric type:  $E(2n + 1)$ , and the amplitude of the transition is proportional to  $(\omega r)^{(2n+1)}$ , while for the latter states the lowest multipole is of the magnetic type:  $M(2n)$ , and the amplitude of the transition is proportional to  $(\omega/m) (\omega r)^{(2n)}$ . The amplitude of the annihilation of a state with orbital momentum  $L$  at distances  $O(m^{-1})$  contains  $L$  derivatives of the wave function at the origin, i.e. it is proportional to  $(\kappa/m)^L$ . Multiplying the indicated factors for both sets of the intermediate states, and taking into account that the  $\exp(-\kappa r)$  behavior of the Green's function constrains the product  $\kappa r$  at order one, one readily finds in both cases that the contribution of the corresponding intermediate state in the amplitude described by Fig.1 contains the factor  $(\omega/m)^{(2n+1)}$ . Thus in the nonrelativistic region of  $\omega \ll m$  it is sufficient to consider only the intermediate states with the lowest  $n$ , i.e.  $n = 0$ . Clearly, these states are the  ${}^1S$  and  ${}^3P$ , and they are reached from the initial  ${}^3S$  state by respectively  $M1$  and  $E1$  radiative transitions. It is important to emphasize that both these intermediate states provide contribution of the same order in the nonrelativistic limit[4, 2], and also to notice that the spatial extent,  $\sim (m\alpha)^{-1}$ , of the initial state of positronium does not enter as a parameter in the discussed multipole expansion<sup>2</sup>

### 3 The Coulomb interaction effect in the spectrum

Using the standard expressions for the Hamiltonian of the  $M1$  and  $E1$  interaction, and also using the well known amplitudes of the two-photon annihilation of the  ${}^1S$  [6] and  ${}^3P$  [7] states, one can write the expression for the differential rate of the o-Ps annihilation in the limit  $x \equiv \omega/m \ll 1$  in the form[2]

$$\frac{d\Gamma}{dx} = \frac{m\alpha^6}{9\pi} x \left[ |a_m|^2 + \frac{7}{3} |a_e|^2 \right] , \quad (1)$$

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<sup>2</sup>The distances in the radiative transition amplitude are constrained by the falloff of the Green's function, rather than by the falloff of the wave function of the initial state. In particular at  $\kappa \gg m\alpha$  the initial state wave function enters only through its value at the origin.

where  $a_m$  and  $a_e$  are respectively the magnetic and electric dipole amplitudes, which can be written in terms of the wave function and of the discussed Coulomb Green's function  $G_c(\mathbf{x}, \mathbf{y}; -\kappa^2/m)$  as follows,

$$a_m = \frac{\omega}{\psi_0(0)} \int G_c(\mathbf{0}, \mathbf{y}; -\kappa^2/m) \psi_0(\mathbf{y}) d^3y , \quad (2)$$

$$a_e = \frac{\omega}{3\psi_0(0)} \int y_i \left[ \frac{\partial}{\partial x_i} G_c(\mathbf{x}, \mathbf{y}; -\kappa^2/m) \right] \Big|_{\mathbf{x}=\mathbf{0}} \psi_0(\mathbf{y}) d^3y . \quad (3)$$

Here  $\psi_0(\mathbf{y})$  stands for the wave function of the initial state of the orthopositronium, and the factor  $\psi_0(0)$  appears in the denominator in both these formulas due to that its value is already included in the normalization in eq.(1). In other words, the amplitudes  $a_m$  and  $a_e$  are normalized in such a way that they both are equal to one if the Coulomb Green's function  $G_c$  is replaced by the free motion one  $G_f$ , in which limit the lowest-order formula[1] for the spectrum is reproduced.

The magnetic amplitude  $a_m$  is however trivial and is equal to one also if the Coulomb interaction is taken into account. This clearly is a consequence of that the spatial wave function of the ground  $^3S$  state is orthogonal to those of all the  $^1S$  states except for the ground one, where the overlap integral is equal to one. The relation  $a_m = 1$  is valid up to relativistic terms, including the  $^3S - ^1S$  hyperfine splitting, which effects are beyond the intended accuracy<sup>3</sup>.

The integral in eq.(3) can be calculated using the partial wave expansion of the Green's function

$$G_c(\mathbf{x}, \mathbf{y}; E) = \sum_{\ell} (2\ell + 1) G_{\ell}(x, y; E) P_{\ell} \left( \frac{\mathbf{x} \cdot \mathbf{y}}{xy} \right) , \quad (4)$$

with  $P_{\ell}(z)$  being the Legendre polynomials, and the following representation[8] of the partial wave Green's functions  $G_{\ell}$  in the Coulomb problem:

$$G_{\ell} \left( x, y; \frac{-\kappa^2}{m} \right) = \frac{m\kappa}{2\pi} (2\kappa x)^{\ell} (2\kappa y)^{\ell} e^{-\kappa(x+y)} \sum_{n=0}^{\infty} \frac{L_n^{(2\ell+1)}(2\kappa x) L_n^{(2\ell+1)}(2\kappa y) n!}{(n + \ell + 1 - \nu)(n + 2\ell + 1)!} , \quad (5)$$

where  $\nu = m\alpha/(2\kappa)$ , and  $L_n^{(p)}(z)$  are the Laguerre polynomials defined as

$$L_n^{(p)}(z) = \frac{e^z z^{-p}}{n!} \left( \frac{d}{dz} \right)^n e^{-z} z^{n+p} . \quad (6)$$

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<sup>3</sup>The hyperfine splitting, becoming essential at very small  $\omega$ :  $\omega \sim m\alpha^4$ , was considered in Ref.[2] in order to establish the low-energy behavior mandated by the Low theorem.

(Usage of this representation can also be found in similar problems in Refs. [4] and [9].)

The expression in eq.(3) contains only the  $P$  wave partial Green's function  $G_1$  and thus can be written as

$$\begin{aligned} a_e(\omega) &= \frac{4\pi\omega}{\psi_0(0)} \int_0^\infty G_1\left(0, y; \frac{-\kappa^2}{m}\right) \psi_0(y) y^4 dy = \\ &= \frac{1-\nu^2}{24} \sum_{n=0}^\infty \frac{1}{n+2-\nu} \int_0^\infty \exp\left(-\frac{1+\nu}{2} z\right) L_n^{(3)}(z) z^4 dz = \\ &= \frac{64}{(1+\nu)^4} \sum_{n=0}^\infty \frac{1}{n+2-\nu} \left[ \frac{(n+4)!}{24 n!} \left(-\frac{1-\nu}{1+\nu}\right)^n - \frac{1+\nu}{2} \frac{(n+3)!}{6 n!} \left(-\frac{1-\nu}{1+\nu}\right)^n \right]. \end{aligned} \quad (7)$$

Here a use is made of the explicit form of the ground state wave function:  $\psi_0(y)/\psi_0(0) = \exp(-m\alpha y/2)$  as well as of the relation  $m\omega/\kappa^2 = 1 - \nu^2$ . In the last transition the formula (6) is used to perform integration by parts. The sum in the latter expression in eq.(7) is of the Gauss hypergeometric type and can be done explicitly in terms of the hypergeometric function  ${}_2F_1$ , so that the final result can be written as

$$a_e(\omega) = \frac{(1-\nu)(3+5\nu)}{3(1+\nu)^2} + \frac{8\nu^2(1-\nu)}{3(2-\nu)(1+\nu)^3} {}_2F_1\left(2-\nu, 1; 3-\nu; -\frac{1-\nu}{1+\nu}\right). \quad (8)$$

The explicit relation between the photon energy  $\omega$  and the Coulomb parameter  $\nu$  reads as

$$\omega = \frac{m\alpha^2}{4} \frac{1-\nu^2}{\nu^2} = \frac{1}{2} \text{Ry} \frac{1-\nu^2}{\nu^2}. \quad (9)$$

The amplitude  $a_e$  given by the formula (8) is fully equivalent to its integral representation described in Ref.[2].

## 4 One-loop QED correction to the photon spectrum at

$$\omega \ll m$$

The result in eq.(8) can be expanded in  $\nu$ . The linear term in this expansion describes the correction of the first order in  $\alpha$ :

$$a_e = 1 - \frac{4}{3} \nu + O(\nu^2) = 1 - \frac{2\alpha}{3} \sqrt{\frac{m}{\omega}} + O(\alpha^2). \quad (10)$$

When used in eq.(1) for the differential decay rate the linear term in this expansion determines the asymptotic behavior of the one-loop QED correction to the spectrum at  $\omega \ll m$ .

However, the slow  $1/\sqrt{x}$  rise of this correction toward small  $x$  makes potentially important the higher terms of the expansion in  $x$  of the  $O(\alpha)$  one-loop correction. Clearly, the relativistic expansion for the discussed terms due to the Coulomb interaction goes in powers of  $\kappa^2/m^2 \approx \omega/m = x$ , so that the next term of this type is proportional to  $\alpha\sqrt{x}$ . The same behavior is true for the correction terms arising from the full relativistic scattering kernel for the initial state and from the process  $(o\text{-Ps}) \rightarrow \gamma^* \rightarrow 3\gamma$ . Although the latter two mechanisms together account for almost 85% of the  $O(\alpha)$  correction to the total rate[10], in the spectrum at low  $x$  their relative contribution starts only as  $\alpha\sqrt{x}$ . Such terms are beyond the scope of this paper.

On the other hand, the constant in  $x$  term, i.e. of order  $\alpha x^0$  at small  $x$ , can be quite readily deduced from the known in the literature results for the one-loop corrections to the total rates of the two photon annihilation of the  $C$ -even  $S$  and  $P$  states of the orthopositronium. These corrections are of genuinely radiative nature and arise from distances of order  $1/m$ , as opposed to the so far discussed Coulomb effects determined by the electron-positron interaction at distances of order  $1/\kappa$ . The sources of such correction terms can be easily identified from the graph of Fig.1. Indeed, the elements of the considered process determined by the distances of order  $1/m$  are the amplitude of the annihilation into two hard photons and the electron magnetic moment entering the Hamiltonian of the  $M1$  interaction. Accordingly, in order to include the discussed radiative corrections, the equation (1) should be rewritten as

$$\frac{d\Gamma}{dx} = \frac{m\alpha^6}{9\pi} x \left\{ |a_m|^2 \left(\frac{g_e}{2}\right)^2 \frac{\Gamma(^1S_0 \rightarrow 2\gamma)}{\Gamma_0(^1S_0 \rightarrow 2\gamma)} + |a_e|^2 \left[ \frac{\Gamma(^3P_0 \rightarrow 2\gamma)}{\Gamma_0(^3P_0 \rightarrow 2\gamma)} + \frac{4}{3} \frac{\Gamma(^3P_2 \rightarrow 2\gamma)}{\Gamma_0(^3P_2 \rightarrow 2\gamma)} \right] \right\}, \quad (11)$$

where  $g_e$  is the electron gyromagnetic ratio, and the ratios of the decay rates for the indicated  $C$ -even states are to their values ( $\Gamma_0$ ) in the lowest order<sup>4</sup>. The  $O(\alpha)$  terms in these ratios are known in the literature:

$$\frac{\Gamma(^1S_0 \rightarrow 2\gamma)}{\Gamma_0(^1S_0 \rightarrow 2\gamma)} = 1 + \frac{\alpha}{\pi} \left( \frac{\pi^2}{4} - 5 \right) \quad (12)$$

for the parapositronium decay[11], and

$$\frac{\Gamma(^3P_0 \rightarrow 2\gamma)}{\Gamma_0(^3P_0 \rightarrow 2\gamma)} = 1 + \frac{\alpha}{\pi} \left( \frac{\pi^2}{4} - \frac{7}{3} \right), \quad \frac{\Gamma(^3P_2 \rightarrow 2\gamma)}{\Gamma_0(^3P_2 \rightarrow 2\gamma)} = 1 - \frac{4\alpha}{\pi} \quad (13)$$

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<sup>4</sup>There is of course no contribution from the  $^3P_1$  state due to the Landau-Yang theorem.

for the  $^3P_{0,2}$  states, which formulas are an adaptation to QED of the quarkonium results for QCD corrections in Ref.[12].

Using the expressions (12) and (13) and also the famous Schwinger's result  $g_e/2 = 1 + \alpha/(2\pi) + O(\alpha^2)$  in eq.(11), the formula for the spectrum can be written including the radiative correction:

$$\begin{aligned} \frac{d\Gamma}{dx} &= \frac{m\alpha^6}{9\pi} x \left\{ |a_m|^2 \left[ 1 + \frac{\alpha}{\pi} \left( \frac{\pi^2}{4} - 4 \right) \right] + |a_e|^2 \left[ \frac{7}{3} + \frac{\alpha}{\pi} \left( \frac{\pi^2}{4} - \frac{23}{3} \right) \right] \right\} \\ &= \frac{10m\alpha^6}{27\pi} x \left[ 1 - \frac{\alpha}{\pi} \left( \frac{14\pi}{15\sqrt{x}} - \frac{3\pi^2}{20} + \frac{7}{2} \right) + O(\alpha^2) \right], \end{aligned} \quad (14)$$

where the last expression also includes the first Coulomb correction to  $a_e$  from eq.(8).

A general calculation of the one-loop QED correction to the spectrum in o-Ps decay has been done numerically by Adkins[3], and it is instructive to compare the two results. For the purpose of such comparison, following the conventions of Ref.[3], we write the formula for the differential probability in the form

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dx} = \gamma_0(x) + \frac{\alpha}{\pi} \gamma_1(x). \quad (15)$$

Here  $\Gamma_0 = 2(\pi^2 - 9)m\alpha^6/(9\pi)$  is the total decay rate in the lowest order[1],  $\gamma_0$  is the normalized differential decay rate in the same lowest order, for which we use here its nonrelativistic limit at small  $x$ :  $\gamma_0(x) = (5/3)x/(\pi^2 - 9)$ , and  $\gamma_1(x)$  is the first-order QED correction to the spectrum. From the equation (14) one readily finds that under the described conventions  $\gamma_1(x)$  is given by

$$\gamma_1(x) = -\frac{1}{9(\pi^2 - 9)} \left[ 14\pi\sqrt{x} + 15 \left( \frac{7}{2} - \frac{3\pi^2}{20} \right) x \right]. \quad (16)$$

In Ref.[3] the physical region of  $x$  from  $x = 0$  to  $x = 1$  is divided into 20 equal bins of 0.05 each, and the integral of  $\gamma_1(x)$  over each of the bins is tabulated. In particular in the first two bins the integrals are found as respectively -0.0502(11) and -0.0994(14). The formula in eq.(16) gives for the same integrals numerical values of -0.0467 and -0.0911. The difference from the result of the full calculation[3], albeit numerical, is in a reasonable agreement with the accuracy expected from using only the first two terms of the expansion in  $\sqrt{x}$  at  $x \approx 0.05 - 0.1$ .



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